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SYMBOLIC-NUMERIC APPROACH FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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Abstract: This paper presents a symbolic-numeric approach for solving linear differential equations of the fourth order in the form of generalized power series. The working program allows to find solutions to differential equations of the fourth order in the form of power series, generally, of any order, but is limited by capabilities of a given computer. Some examples of solving differential equations of the fourth order are presented, which show the efficiency of the developed program. The results are consistent with the available literature data.

Keywords: differential equations of the fourth order; generalized power series; singular regular points.

I. INTRODUCTION

An effective method for integrating differential equations is finding solutions in the form of power series and the subsequent finding of the coefficients of these series [1-4]. But in practice, with concrete calculations, one has to deal with a large amount of calculations when finding unknown coefficients of power series, and the complexity of finding solutions increases in cases when peculiarities appear. However, the use of modern computers with software packages for analytical calculations, such as Maple, Mathematica, Reduce, and others [5-7], allows us to quickly perform the necessary calculations to build solutions of linear differential equations in the form of series, and to very large orders.

In problems of mathematical physics, linear differential equations of second and fourth orders most often arise [8]. It should also be noted that the solutions of some nonlinear differential equations can be expressed in terms of linearly independent solutions of the corresponding linear differential equations. For example, the solution of the non-linear Ermakov-Milne-Pinney equation is constructed from two independent solutions of a linear second-order ordinary differential equation.

The paper presents a computational scheme for solving fourth-order ordinary linear differential equations in the form of generalized power series using the Maple computer algebra system. Also, using the compiled program, linearly independent solutions were found for a number of specific differential equations.

II. METHOD

Computational Scheme

$$y^{(IV)}(x) + P_3(x)\,y'''(x) + P_2(x)\,y''(x) + P_1(x)\,y'(x) + P_0(x)\,y(x) = 0 \qquad (1)$$

If coefficients-functions $P_3(x)$, $P_2(x)$, $P_1(x)$, $P_0(x)$ o not contain singular regular points and are holomorphic functions in the vicinity of point $x=x_0$, i.e. have the following forms:

$$P_0(x)=\sum_{k=0}^{\infty} p_k^{(0)}(x-x_0)^k ,$$

$$P_1(x)=\sum_{k=0}^{\infty} p_k^{(1)}(x-x_0)^k$$

$$P_2(x)=\sum_{k=0}^{\infty} p_k^{(2)}(x-x_0)^k ,$$

$$P_3(x)=\sum_{k=0}^{\infty} p_k^{(3)}(x-x_0)^k$$

then four linear independent solutions y_1 , y_2 , y_3 and y_4 can be presented in the form of following power series:

$$y_1(x)=1+\sum_{k=2}^{\infty} c_k^{(1)}(x-x_0)^k ,$$
$$y_2(x)=x-x_0+\sum_{k=2}^{\infty} c_k^{(2)}(x-x_0)^k , \qquad (3)$$
$$y_3(x)=(x-x_0)^2/2+\sum_{k=2}^{\infty} c_k^{(3)}(x-x_0)^k ,$$
$$y_4(x)=(x-x_0)^3/3+\sum_{k=2}^{\infty} c_k^{(4)}(x-x_0)^k$$

Coefficients $c_k^{(1)}$, $c_k^{(2)}$, $c_k^{(3)}$, $c_k^{(4)}$ are defined uniquely by means of substitution of series (3) in the equation (1) and equating with zero coefficients at various orders of an independent variable in the left part of the received equality.

In the presence of poles the type that not be higher than the fourth order in a point $x=x_0$ then solutions (3) to be other and depending on roots of the defining equation than the fourth (see, for example, [9-11]). From the theory of the ordinary differential equations [9] it is known that in order that the equation, in particular, of a look (1) had in the neighborhood of a special point at least if only one partial solution in the form of the generalized power series.

$$y(x)=(x-x_0)^{\rho}\sum_{k=0}^{\infty} c_k(x-x_0)^k, \quad (c_0\neq 0) \qquad (4)$$

where the indicator ρ is some constant number it is, enough, that this equation had an appearance

$$y^{(IV)}(x)+\frac{\sum_{k=0}^{\infty} p_k^{(3)}}{x-x_0}y'''(x)+\frac{\sum_{k=0}^{\infty} p_k^{(2)}}{(x-x_0)^2}y''(x)+$$
$$\frac{\sum_{k=0}^{\infty} p_k^{(1)}}{(x-x_0)^3}y'(x)+\frac{\sum_{k=0}^{\infty} p_k^{(0)}}{(x-x_0)^4}y(x)=0. \qquad (5)$$

The indicator ρ is found from the so-called defining equation:

$$\rho(\rho-1)(\rho-2)(\rho-3)+\rho(\rho-1)(\rho-2)\,p_0^3+$$
$$\rho(\rho-1)\,p_0^2+\rho\,p_0^1+p_0^0=0 \qquad (6)$$

Let us assume, there are also roots ρ_1 , ρ_2 , ρ_3 and ρ_4 of the equation (6). Then, if roots of the defining equation, are also independent, and any two of them don't differ on an integer, then to each number there corresponds a certain sequence of coefficients, and all four independent solutions forming fundamental system turn out are equal to

$$y_1(x)=(x-x_0)^{\rho_1}\sum_{k=0}^{\infty} c_k^{(1)}(x-x_0)^k, \quad (c_0^{(1)}\neq 0),$$
$$y_2(x)=(x-x_0)^{\rho_2}\sum_{k=0}^{\infty} c_k^{(2)}(x-x_0)^k, \quad (c_0^{(2)}\neq 0), \qquad (7)$$
$$y_3(x)=(x-x_0)^{\rho_3}\sum_{k=0}^{\infty} c_k^{(3)}(x-x_0)^k, \quad (c_0^{(3)}\neq 0)$$
$$y_4(x)=(x-x_0)^{\rho_4}\sum_{k=0}^{\infty} c_k^{(4)}(x-x_0)^k, \quad (c_0^{(4)}\neq 0)$$

Coefficients $c_k^1, c_k^2, c_k^3, c_k^4$, are also defined by substitution of ranks (7) in the equation (5), at the same time coefficients c_0^1, c_0^2, c_0^3, c_0^4, and remain any (further we will put their equal to unit). These last coefficients are defined by initial conditions.

If the founded four values ρ are such that two or several differ on an integer, then they can be located in the form of the following independent subsequences:

$$\rho_1,\rho_2,\ldots,\rho_{\alpha-1},$$
$$\rho_\alpha,\rho_{\alpha+1},\ldots,\rho_{\beta-1} \qquad (8)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

so that values in each sequence of the various were only integers, and the real part of the subsequences would be a non-increasing sequence. Only the first member of each sequence gives the solution (4), since, for example, any

member of the sequence is equal to or less it than on the positive integer.
Let's consider one of the sequences of indicators of the differential equation, for example, the sequence

$$\rho_1, \rho_2, \ldots, \rho_{\alpha-1},$$

which is so located that if $\chi < \lambda$, then $\rho_\chi - \rho_\lambda$ a positive integer or zero. As these indicators aren't surely equal, they can be divided into subsequences so that members of each subsequence were equal among themselves. So, suppose, that $\rho_1 = \rho_2 = \ldots = \rho_{i-1}$ correspond to a multiple root i ; $\rho_i = \rho_{i+1} = \ldots = \rho_{j-1}$ correspond to a multiple root ; correspond to a multiple root $j-i$, $\rho_j = \rho_{j+1} = \ldots = \rho_{k-1}$ etc. until a row isn't exhausted.
Let's consider an indicator ρ_1 in the first subsequence. In this case arises subsequence of solutions:

$$y_1(x) = w_1(x,\rho_1)$$
$$y_2(x) = w_1(x,\rho_1)\ln x + w_2(x,\rho_1) \qquad (9)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_{i-1}(x) = w_1(x,\rho_1)(\ln x)^{i-1} + (i-1)w_2(x,\rho_1)(\ln x)^{i-2} + \ldots$$
$$y_{i-1}(x) = w_1(x,\rho_1)(\ln x)^{i-1} + (i-1)w_2(x,\rho_1)(\ln x)^{i-2} + \ldots$$

where $w_s(x,\rho_s) = (x-x_0)^{\rho_s}\sum_{k=0}^{\infty} c_k^s (x-x_0)^k, (c_0^s \neq 0, s = 1,2,3,\ldots)$,
the presence of $w_1(x,\rho_1)(\ln x)^{r-1}$ terms in W_r shows that all these solutions are linearly-independent.

Let's consider r_i an indicator in the second subsequence, to it corresponds $j-i$ solutions:

$$y_i(x) = w_1(x,\rho_i)(\ln x)^i + i\, w_2(x,\rho_i)(\ln x)^{i-1} + \ldots + w_i(x,\rho_i)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (10)$$
$$y_{j-1}(x) = w_1(x,\rho_i)(\ln x)^{j-1} + (j-1)w_2(x,\rho_i)(\ln x)^{j-2} + \ldots + w_{j-1}(x,\rho_i)$$

Similarly, the subsequence with index j gives $k-j$ solutions, etc. until the all roots in subsequence is exhausted. Since the functions $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ are linearly independent solutions of equation (5), then with their help, we find the general solution:

$$y(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + C_3 \cdot y_3(x) + C_4 \cdot y_4(x) \qquad (11)$$

Examples of Solving Differential Equations of the Fourth Order

The developed program [12] allows to find solutions of the differential equations of the fourth order in the form of power series, generally, of any order, but is limited by capabilities of a given computer. The program was used to make symbolical and numerical calculations for some differential equations, the results of which exactly coincide with analytical solutions given.

Example 1. If the coefficients-function have values $P_3(x) = 0$, $P_2(x) = 0$, $P_1(x) = 0$, $P_0(x) = 4$ equation (1) takes the following form

$$y^{(IV)}(x) + 4y(x) = 0$$

which also has no singular points.
Developed with the help of a program [12] receive the following four linearly independent solutions:

$$y_1(x) = 1 - \frac{1}{6}x^4 + \frac{1}{2520}x^8 - \frac{1}{7484400}x^{12} + \frac{1}{81729648000}x^{16} + \ldots$$
$$y_2(x) = x - \frac{1}{30}x^5 + \frac{1}{22680}x^9 - \frac{1}{97297200}x^{13} + \frac{1}{1389404016000}x^{17} + \ldots$$
$$y_3(x) = \frac{1}{2}x^2 - \frac{1}{180}x^6 + \frac{1}{226800}x^{10} - \frac{1}{1362160800}x^{14} + \frac{1}{2500972288000}x^{18} + \ldots$$
$$y_4(x) = \frac{1}{3}x^3 - \frac{1}{630}x^7 + \frac{1}{1247400}x^{11} - \frac{1}{10216206000}x^{15}$$

which exactly coincide with the analytical solutions:
$y_1(x) = e^x \sin(x)$, $y_2(x) = e^x \cos(x)$,
$y_3(x) = e^{-x}\sin(x)$, $y_4(x) = e^{-x}\cos(x)$.

Example 2. For a differential equation
$$y^{(IV)}(x) + y'''(x) - 3y''(x) - 5y'(x) - 2y(x) = 0$$

which has no singularities, but with multiple roots in the characteristic equation of multiplicity 3 and the fourth root $k_1 = k_2 = k_3 = -1$. Was obtained the following four $k_4 = 2$ linearly independent solutions:

$$y_1(x)=1+\frac{1}{12}x^4-\frac{1}{60}x^5+\frac{1}{90}x^6-\frac{1}{1260}x^7+\frac{11}{20160}x^8-\frac{1}{181440}x^9+\frac{1}{60480}x^{10}+...$$

$$y_2(x)=x+\frac{5}{24}x^4-\frac{1}{40}x^5+\frac{1}{40}x^6-\frac{1}{2520}x^7+\frac{17}{13440}x^8-\frac{1}{13440}x^9+\frac{19}{453600}x^{10}+...$$

$$y_3(x)=\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{60}x^5+\frac{1}{80}x^6+\frac{1}{420}x^7+\frac{31}{40320}x^8+\frac{1}{6720}x^9+\frac{13}{403200}x^{10}+...$$

$$y_4(x)=\frac{1}{3}x^3-\frac{1}{12}x^4+\frac{1}{15}x^5-\frac{1}{180}x^6+\frac{11}{2520}x^7+\frac{1}{20160}x^8+\frac{1}{6048}x^9+\frac{1}{75600}x^{10}+...$$

which exactly coincide with the analytical solution $y_1(x)=e^{2x}$, $y_2(x)=e^{-x}$, $y_3(x)=x\,e^{-x}$, $y_4(x)=x^2e^{-x}$.

Example 3. If the coefficients of the function have values $P_3(x)=4$, $P_2(x)=6$, $P_1(x)=4$, $P_0(x)=1$, then equation (1) will be as follows

$$y^{(IV)}(x)+4y'''(x)+6y''(x)+4y'(x)+y(x)=0$$

This equation has no singular points, but with multiple roots in the defining equation are equal of multiplicity 4. Using the developed program, we find four linearly $k_1=k_2=k_3=k_4=-1$ independent solutions:

$$y_1(x)=1-\frac{1}{24}x^4+\frac{1}{30}x^5-\frac{1}{72}x^6+\frac{1}{252}x^7-\frac{1}{1152}x^8+\frac{1}{6480}x^9-\frac{1}{43200}x^{10}+...$$

$$y_2(x)=x-\frac{1}{6}x^4+\frac{1}{8}x^5-\frac{1}{20}x^6+\frac{1}{72}x^7-\frac{1}{336}x^8+\frac{1}{1920}x^9-\frac{1}{12960}x^{10}+...$$

$$y_3(x)=\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{1}{6}x^5-\frac{1}{16}x^6+\frac{1}{60}x^7-\frac{1}{288}x^8+\frac{1}{1680}x^9-\frac{1}{11520}x^{10}+...$$

$$y_4(x)=\frac{1}{3}x^3-\frac{1}{3}x^4+\frac{1}{6}x^5-\frac{1}{18}x^6+\frac{1}{72}x^7-\frac{1}{360}x^8+\frac{1}{2160}x^9-\frac{1}{15120}x^{10}+...$$

which exactly coincide with the analytical solutions $y_1(x)=e^{-x}$, $y_2(x)=xe^{-x}$, $y_3(x)=x^2e^{-x}$, $y_4(x)=x^3e^{-x}$.

Example 4. Consider the equation

$$x^4y^{(IV)}(x)+x^2y''(x)-xy'(x)=0$$

This equation has a singular point $x_0=0$. In this case, the defining equation has roots equal $\rho_1=\rho_2=\rho_3=2$, $\rho_4=0$. Using the developed program [12], obtained the four linearly independent solutions: $y_1(x)=x^2$, $y_2(x)=x^2\ln(x)+x^2$, $y_3(x)=x^2\ln^2(x)+2x^2\ln(x)+x^2$, $y_4(x)=1$.

Example 5. Consider the equation

$$x^4y^{(IV)}(x)+2x^3y'''(x)+x^2y''(x)-xy'(x)+y(x)=0$$

This equation has a singular point $x_0=0$. In this case the defining equation has roots equal $\rho_1=\rho_2=\rho_3=\rho_4=1$. Using the developed program [12], obtained the four linearly independent solutions:

$y_1(x)=x$, $y_2(x)=x\ln(x)+x$,

$y_3(x)=x\ln^2(x)+2x\ln(x)+x$,

$y_4(x)=x\ln^3(x)+3x\ln^2(x)+2x\ln(x)+x$

With the help of the developed program, solutions of two problems from the theory of resistance of materials were found.

Example 6. The basic equation of stability for a flat bend shape of an I-beam, to which a bending moment and external force is applied, is a fourth-order linear differential equation [4].

$$Q^{(IV)}(x)-\frac{1}{a^2}Q''(x)-\frac{1}{d^4}Q(x)=0$$

where a, d - geometric and strength characteristics of the beam, $Q(x)$ - node twisting the beam along its length. In this equation, the coefficients are functions $P_3(x)=0$, $P_2(x)=-\frac{1}{a^2}$, $P_1(x)=0$, $P_0(x)=-\frac{1}{d^4}$, have no particular points.

Using the developed program [12], we obtain four linearly independent solutions:

$$Q_1(x)=1+\frac{1}{24d^4}x^4+\frac{1}{720a^2d^4}x^6+\frac{(d^4+a^4)}{40320a^4d^8}x^8+\frac{(d^4+2a^4)}{3628800a^6d^8}x^{10}+\frac{(d^8+3a^4d^4+a^8)}{479001600a^8d^{12}}x^{12}$$

$$Q_2(x)=x+\frac{1}{120d^4}x^5+\frac{1}{5040a^2d^4}x^7+\frac{(d^4+a^4)}{362880a^4d^8}x^9+\frac{(d^4+2a^4)}{39916800a^6d^8}x^{11}+$$

$$+\frac{(d^8+3a^4d^4+a^8)}{6227020800a^8d^{12}}x^{13}$$

$$Q_3(x)=\frac{x^2}{2}+\frac{1}{24a^2}x^4+\frac{(d^4+a^4)}{720a^4d^4}x^6+\frac{(d^4+2a^4)}{40320a^6d^4}x^8+\frac{(d^8+3a^4d^4+a^8)}{3628800a^8d^8}x^{10}+$$

$$+\frac{(d^8+4a^4d^4+3a^8)}{479001600a^{10}d^8}x^{12}$$

$$Q_4(x)=\frac{x^3}{3}+\frac{1}{60a^2}x^5+\frac{(d^4+a^4)}{2520a^4d^4}x^7+\frac{(d^4+2a^4)}{181440a^6d^4}x^9+\frac{(d^8+3a^4d^4+a^8)}{19958400a^8d^8}x^{11}+$$
$$+\frac{(d^8+4a^4d^4+3a^8)}{3113510400a^{10}d^8}x^{13}$$

III. CONCLUSION

This paper presents a computational scheme for solving fourth-order ordinary differential equations in the form of generalized power series. The Maple program was composed based on the developed algorithm and some problems from the material resistance theory were solved. Moreover, some examples of solving differential equations of the fourth order are presented, which show the efficiency of the developed program. The results are consistent with the available literature data.

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